of customers who show up out of θ reservations, in which case

$$X(\theta) = \sum_{i=1}^{\theta} Y_i,$$

where Y_i are i.i.d. Bernoulli random variables with $P(Y_i = 1) = q$ and $P(Y_i = 0) = 1 - q$.

Given a function $g(x)$, suppose we are interested in determining properties of the expected value $E[g(X(\theta)]$ as a function of θ. For example, if g is increasing in x, is $E[g(X(\theta)]$ increasing in θ? If g is convex in x, is $E[g(X(\theta)]$ convex in θ? Stochastic monotonicity and convexity identify classes of random variables $X(\theta)$ for which such statements can be made. A good source for this material is the series of papers by Shaked and Shantikumar [460, 461] and their subsequent book [462].

DEFINITION B.1 *The random function* $X(\theta)$ *is* **stochastically increasing** *in* θ *if for all* $\theta_1 \geq \theta_2$, $P(X(\theta_1) > x) \geq P(X(\theta_2) > x)$.

A random function $X(\theta)$ is *stochastically decreasing* in θ if—$X(\theta)$ is stochastically increasing. An equivalent definition is provided by the following proposition:

PROPOSITION B.1 $X(\theta)$ *is stochastically increasing in* θ *if for any* $\theta_1 \geq \theta_2$, *there exists two random variables* X_1 *and* X_2 *defined on a common probability space* $(\Omega, \mathcal{F}, P)$, *such that* X_1 *and* X_2 *are equal in distribution to* $X(\theta_1)$ *and* $X(\theta_2)$ *(respectively), and they satisfy* $X_1(\omega) \geq X_2(\omega)$ *for all* $\omega \in \Omega$.

Continuing our example, we see that if $X(\theta) = \sum_{i=1}^{\theta} Y_i$, where Y_i are i.i.d. Bernoulli random variables, then $X(\theta)$ is stochastically increasing, since we can consider ω to define an infinite sequence $\{Y_1, Y_2, \ldots\}$ and consider $X(\theta)$ to be the sum of the first θ variables in this sequence. For every $\theta_1 \geq \theta_2$, the sums $X(\theta_1)$ and $X(\theta_2)$ will have the required distribution and $X(\theta_1) \geq X(\theta_2)$ for every such sequence ω.

The following proposition follows easily from this sample path definition of monotonicity:

PROPOSITION B.2 $X(\theta)$ is *stochastically increasing in* θ *if and only if for any real valued, increasing function* $g(x)$, $E[g(X(\theta)]$ *is increasing in* θ.

Similarly, one can define a notion of stochastic convexity for $X(\theta)$:

DEFINITION B.2 $X(\theta)$ *is* **stochastically convex** *(SCX) if for any real valued, convex function* $g(x)$, $E[g(X(\theta)]$ *is convex in* θ.

We say $X(\theta)$ is *stochastically concave* (SCV) if $-X(\theta)$ is stochastically convex, and we say $X(\theta)$ is *stochastically linear* if it is both stochastically convex and stochastically concave.

To verify whether the above holds is often difficult. However, two stronger notions of stochastic convexity are quite useful and both imply stochastic convexity. These are:

DEFINITION B.3 $X(\theta)$ *is said to be* **strongly stochastically convex** *(SSCX) if* $X(\theta) = \psi(Z, \theta)$ *where Z is a random variable independent of* θ *and* ψ *is convex in* θ *for every value of Z.*